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A GENERALIZATION OF LAYMAN–LOTOCKIĬ EXPONENTIAL INTERPOLATION

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Abstract—For equally spaced data points $f(i)$, $i = 0(1)n$, an exponential interpolation polynomial for $f(z)$, namely $\Sigma_{i=0}^n a_i r_i^z$, where r_i are arbitrary quantities, is obtainable formally from the Newton divided difference series for the displacement operator $E^z = (1+\Delta)^z$, regarding Δ as the variable, for $\Delta = s_i = r_i - 1$. This generalization of Layman–Lotockiĭ exponential interpolation in which $r_i = i + 1$, employs "general factorial differences" $\Pi_{j=0}^i(\Delta - s_j)f(0)$, $i = 0(1)\ n-1$. For the confluent case, where $f^{(i)}(0)$ is given instead of $f(i)$, $i = 0(1)n$, the results are similar, the derivative operator D replacing the Δ.

INTRODUCTION

Layman[1–4] and Lotockiĭ[5], working apparently independently around the same time, studied this interpolation series for $f(z)$ given at equally spaced arguments which are transformable into 0, 1, 2, . . . ,

$$f(z) \sim \sum_{i=0}^{\infty} \binom{\Delta}{i} f(0) e_i(z), \tag{1}$$

where $\binom{\Delta}{i} f(0)$ is proportional to the ith "factorial difference" $\Delta(\Delta - 1)\ldots(\Delta - i + 1)f(0)$, and where

$$e_i(z) = \sum_{j=0}^{i} (-1)^j \binom{i}{j} (i + 1 - j)^z, \tag{2}$$

is the ith advancing difference at $\Delta = 0$ of the displacement operator $E^z = (1+\Delta)^z$ in which Δ, instead of z, is regarded as the variable. From (1) and (2) we obtain for $f(z)$ an $(n + 1)$-point exponential interpolation polynomial of the form $\Sigma_{i=0}^n a_i(i+1)^z$.

We consider here the more general problem of finding the series corresponding to (1), (2), for equally spaced data points $f(i)$, $i = 0, 1, 2, \ldots,$ which yields an $(n + 1)$-point exponential interpolation polynomial of the form $\Sigma_{i=0}^n a_i r_i^z$ in which the r_i's are arbitrary quantities, not necessarily real, even though $r_i > 0$ is usually specified when $f(i)$ is real. Layman saw nothing better than Cramer's rule for finding $\Sigma_{i=0}^n a_i r_i^z$ ([1], p. 118). We show here, by the appropriate expansion of the operator $(1 + \Delta)^z$, how to produce the series which generalizes (1), (2), and from which we obtain $\Sigma_{i=0}^n a_i r_i^z$ directly and much more expediently than by Cramer's rule.

We also note briefly the confluent case, where we are given $f^{(i)}(0)$ instead of $f(i)$, $i = 0, 1, \ldots,$ from which we derive similar exponential series involving the derivative operator D instead of Δ, from e^{Dz} in place of $(1 + \Delta)^z$.

Convergence of the interpolation series (3), and (10) below, are still open problems, approaches to which might be suggested by results of Layman and Lotockiĭ concerning the convergence of $\Sigma_{i=0}^{\infty} a_i e_i(z)/i!$ for a given sequence of a_i's and the representability of a given entire function $f(z)$ by the series in (1)[3–5]. For example, in connection with representability there is Layman's theorem on the convergence of (1) for $f(z)$ an entire function of exponential type, provided that a certain set $D(f)$ (see [3], p. 519 for a full description) lies within some suitably defined strip of width π ([3], p. 521).

INTERPOLATION SERIES

Regarding the operator $(1 + \Delta)^z$ as a function of Δ, with z as a parameter, expand it as a Newton divided difference series, for $\Delta = s_i = r_i - 1$, $i = 0, 1, 2, \ldots,$ and apply it to $f(z)$

at $z = 0$, obtaining this formal infinite series:

$$\begin{aligned} f(z) = E^z f(0) = (1+\Delta)^z f(0) &= [s_0]f(0) + [s_0 s_1](\Delta - s_0)f(0) + [s_0 s_1 s_2](\Delta - s_0)(\Delta - s_1)f(0) \\ &\quad + \cdots + [s_0 s_1 \ldots s_i](\Delta - s_0)(\Delta - s_1)\ldots(\Delta - s_{i-1})f(0) \\ &\quad + \cdots, \end{aligned} \tag{3}$$

in which the divided differences $[s_0] = r_0^z$ and

$$[s_0 s_1 \ldots s_i] = \sum_{j=0}^{i} r_j^z / \Pi_{k=0, k\neq j}^{i}(s_j - s_k). \tag{4}$$

The operator $\Pi_{k=0}^{i-1}(\Delta - s_k)$ may be defined as the "ith general factorial difference operator". Notice that

$$\{\Pi_{k=0}^{n}(\Delta - s_k)\} \sum_{i=0}^{n} a_i r_i^z \equiv 0, \tag{5}$$

for all values of z, and Δ for a unit interval in z, i.e. whenever $f(z) \equiv \Sigma_{i=0}^{n} a_i r_i^z$, its $(n+1)$th general factorial difference vanishes identically. The proof of (5) is immediate from the annihilation of $a_k r_k^z$ by the operational factor $\Delta - s_k$. Thus (5) is analogous to the vanishing of the $(n+1)$th difference of any nth degree polynomial, and it may be employed in a similar checking capacity.

When $f(z) = \Sigma_{i=0}^{n} a_i r_i^z$ and the r_i's are not known initially, they can be found from just $2n+2$ data points by the well-known Prony method.

When $z = n$ is within a region of convergence to $f(z)$ of the infinite series in (3), which breaks off after the $(n+1)$th term when $z = n$ (since $[s_0 s_1 \ldots s_i]$ in (4) is then the ith divided difference of the polynomial $(1+\Delta)^n$, which vanishes for any choice of $s_0, s_1, \ldots, s_i$, whenever $i > n$), the right member of (3) gives $f(n)$ for $z = n$. However, even when the infinite series in (3) may not converge to $f(z)$, it is still true that the first $n+1$ terms gives an exponential polynomial $\Sigma_{i=0}^{n} a_i r_i^z$ which equals $f(i)$ when $z = i$, $i = 0(1)n$ (fundamental interpolation property). This follows because, for $z = i$, (a) the $(i+2)$th through the $(n+1)$th term's of (3) vanish, being proportional to the $(i+j)$th divided difference $[s_0 s_1 \ldots s_{i+j}]$, $j = 1(1)n - i$, of $(1+\Delta)^i$, and (b) the first $i+1$ terms are the result of an operator in Δ of the ith degree acting on $f(0)$, that operator (being any of the possible divided difference expansions in Δ-as-variable, of the ith degree polynomial $(1+\Delta)^i$, which makes it equal to $(1+\Delta)^i$ for all numerical values of Δ), having to be identical with $(1+\Delta)^i$ as a polynomial function of Δ, from which we have $(1+\Delta)^i f(0) = E^i f(0) = f(i)$. q.e.d. There is a slight subtlety in the preceding argument that goes from operator to polynomial and finally back to operator.

The uniqueness of the $(n+1)$-point exponential interpolation polynomial $\Sigma_{i=0}^{n} a_i r_i^z$ follows from the non-vanishing $(n+1)$th order determinant $\|r_i^j\|$, $i, j = 0(1)n$, which is a transposed Vandermondian. Thus the series (3), when truncated after the $(n+1)$th term, reproduces any $f(z) \equiv \Sigma_{i=0}^{n} a_i r_i^z$ from its values at $z = 0(1)n$.

From (3) and (4) we obtain this explicit expression for the a_i in the interpolating $\Sigma_{i=0}^{n} a_i r_i^z$ (definitely preferable to Cramer's rule, when n is large):

$$a_i = \left\{ \sum_{j=i}^{n} [\Pi_{l=0}^{j-1}(\Delta - s_l)/\Pi_{l=0, l\neq i}^{j}(s_i - s_l)] \right\} f(0), \tag{6}$$

in which, for a_0, for $j = 0$, $\Pi_{l=0}^{-1}(\Delta - s_l)/\Pi_{l=0, l\neq 0}^{0}(s_0 - s_l) \equiv 1$.

To illustrate (6), we find the coefficients a_i, $i = 0(1)10$, in $f(z) = a_0\ 1.2^z + a_1\ 1.7^z + a_2\ 2.4^z + a_3\ 2.5^z + a_4\ 3^z + a_5\ 3.4^z + a_6\ 3.7^z + a_7\ 3.9^z + a_8\ 4.7^z + a_9\ 5.1^z + a_{10}\ 5.6^z$, having been given $f(z)$ to $10D$ for $z = 0(1)10$ (Table 1A, 1st column). The first ten general factorial differences, computed from the rounded $f(z)$'s (Table 1A, 2nd–11th columns) are all employed in obtaining the first row of Table 1A, the only one needed for (6). The size of $s_i = r_i - 1$, $i = 0(1)10$, increases the errors in successive columns, an *U.B.* for which was derived from the expansion of the operator $\Pi_{l=0}^{j-1}(\Delta - s_l)$. The denominators in (6), $\Pi_{l=0, l\neq i}^{j}(s_i - s_l)$ (Table 1B), magnify the errors when the r_i's are close together. The a_i's found from (6) deviated from the true values of $a_i = (-1)^i + 3/(i+1)$ that were employed in the precomputation of $f(z)$, by somewhat more than 50% of the $U.B.$ Considering that the system of 11 linear equations for the a_i's is not too well-conditioned, the results, shown in Table 1C, might be regarded as satisfactory.

An equivalent alternative form for a_i, which is obtainable directly from the Lagrange interpolation polynomial form of the operator $(1+\Delta)^z$, is not recommended because it requires $n+1$ sets of general factorial differences instead of just the single set in (6).

The interpolating $\Sigma_{i=0}^{n} a_i r_i^z$ is expressible also in this $(n+1)$-point "Lagrangian exponential interpolation polynomial" form:

$$\sum_{i=0}^{n} a_i r_i^z = \sum_{i=0}^{n} L_i^{(n)}(z) f(i), \tag{7}$$

where, from (3) and (4), noting that $\Delta - s_k = E - r_k$ and that $f(i)$ first occurs in the $(i+1)$th term of (3), we have

$$L_i^{(n)}(z) = \sum_{j=i}^{n} \sigma_{i,j}[s_0 \ldots s_j], \tag{8}$$

in which $\sigma_{i,j} = (-1)^{j-1} \times$ that elementary symmetric function of $r_0, \ldots, r_{j-1}$ which is of the $(j-i)$th degree.

Table 1

1A. $f(z)$ and general factorial differences

z	$f(z)$	$(\Delta - s_0)f(z)$	$(\Delta - s_1)(\Delta - s_0)f(z)$
0	10.05963 20346	13.52559 52382	33.28235 71427
1	25.59715 36797	56.27586 90476	160.97162 14287
2	86.99245 34632	256.64059 88096	814.18676 92855
3	361.03154 29654	1250.47578 72618	4246.04555 33573
4	1683.71363 88203	6371.85439 17024	22609.40583 91213
5	8392.31075 82868	33441.55830 50154	1 22134.97448 78195
6	43512.33121 49596	1 78985.62360 63457	6 66508.71172 98481
7	2 31200.42106 42972	9 70784.27186 06359	36 64110.49860 64295
8	12 48224.77713 77925	53 14443.76076 95105	202 53280.98589 84190
9	68 12313.49333 48615	292 87835.37920 65868	
10	374 62611.57120 84206		
$U.B.$ for |error|	$\frac{1}{2} \times 10^{-10}$	1.1×10^{-10}	3×10^{-10}

z	$(\Delta - s_2)\cdots(\Delta - s_0)f(z)$	$(\Delta - s_3)\cdots(\Delta - s_0)f(z)$	$(\Delta - s_4)\cdots(\Delta - s_0)f(z)$
0	81.09396 42862	225.11996 71411	547.00021 10073
1	427.85487 78566	1222.36011 24306	3021.82290 60917
2	2291.99730 70721	6688.90324 33835	16758.44946 61184
3	12418.89651 10638	36825.15919 62689	93228.29418 54536
4	67872.40047 39284	2 03703.77177 42603	5 19916.34273 43099
5	3 73384.77295 90813	11 31027.65805 70908	29 05108.83893 47305
6	20 64489.59045 47941	62 98191.81310 60029	
7	114 59415.78924 29882		
U.B. for |error|	10^{-9}	3.5×10^{-9}	1.4×10^{-8}

z	$(\Delta - s_5)\cdots(\Delta - s_0)f(z)$	$(\Delta - s_6)\cdots(\Delta - s_0)f(z)$	$(\Delta - s_7)\cdots(\Delta - s_0)f(z)$
0	1162.02218 86669	2184.76948 73391	3737.23413 40241
1	6484.25158 54066	12257.83513 46466	21011.19127 62373
2	36249.56600 06510	68816.74830 13590	1 18129.42799 88362
3	2 02940.14250 37677	3 86514.74637 41363	
4	11 37393.27363 80768		
U.B. for |error|	6.22×10^{-8}	2.92×10^{-7}	1.43×10^{-6}

z	$(\Delta - s_8)\cdots(\Delta - s_0)f(z)$	$(\Delta - s_9)\cdots(\Delta - s_0)f(z)$
0	3446.19084 63240	1801.25568 42685
1	19376.82900 05209	
U.B. for |error|	8.17×10^{-6}	4.98×10^{-5}

Table 1 (*contd.*)

1B. $\Pi^{j}_{l=0, l\neq i}(s_i - s_l)$

i \ $j =$	1	2	3	4	5	6	7	8	9	10
0	−0.5	0.6	−0.78	1.404	−3.0888	7.722	−20.8494	72.9729	−284.59431	1252.214964
1	0.5	−0.35	0.28	−0.364	0.6188	−1.2376	2.72272	−8.16816	27.771744	−108.3098016
2		0.84	−0.084	0.0504	−0.0504	0.06552	−0.09828	0.226044	−0.6103188	1.95302016
3			0.104	−0.052	0.0468	−0.05616	0.078624	−0.1729728	0.44972928	−1.394160768
4				0.702	−0.2808	0.19656	−0.176904	0.3007368	−0.63154728	1.642022928
5					1.3464	−0.40392	0.20196	−0.262548	0.4463316	−0.98192952
6						1.638	−0.3276	0.3276	−0.45864	0.871416
7							1.12266	−0.898128	1.0777536	−1.83218112
8								93.93384	−37.573536	33.8161824
9									223.315118208	−111.657559104
10										1415.272345344

1C. $a_i, i = 0(1)10$

i	0	1	2	3	4
a_i from (6)	3.99999 99341	0.50000 062	1.99997 24	−0.24996 04	1.60001 41
true value	4	0.5	2	−0.25	1.6
|error|	6.59×10^{-8}	6.2×10^{-7}	2.76×10^{-5}	3.96×10^{-5}	1.41×10^{-5}

5	6	7	8	9	10
−0.49995 74	1.428526	−0.62497 947	1.33333 2376	−0.699999 729	1.272727 253
−0.5 .	1.428571...	−0.625	1.33333 3333...	0.7	1.272727 2727...
4.26×10^{-5}	4.5×10^{-5}	2.05×10^{-5}	10^{-6}	2.71×10^{-7}	2×10^{-8}

FACTORIAL DERIVATIVE SERIES

Confluent exponential interpolation series, similar to (3), (4) but employing derivatives D^j instead of differences, are derivable formally from the expansion of the operator $E^z = e^{Dz}$, regarding D as the variable. This applies when, in place of equally spaced data points, we have $D^j f(0) \equiv f^{(j)}(0)$, $j = 0, 1, 2, \ldots$.

The Gregory–Newton expansion of e^{Dz}, for $D = 0, h, 2h, \ldots$ gives

$$f(z) = e^{Dz}f(0) = \left\{1 + \left(\frac{e^{hz}-1}{h}\right)D + \left(\frac{e^{hz}-1}{h}\right)^2 D(D-h)/2 + \cdots \right. \\ \left. + \left(\frac{e^{hz}-1}{h}\right)^i D(D-h)\ldots(D-(i-1)h)/i! + \cdots\right\} f(0), \qquad (9)$$

which converts a power series in z to one in $[(e^{hz}-1)/h]$, with coefficients proportional to the "factorial derivatives" $D(D-h)\ldots(D-(i-1)h)f(0)$. The series in (9) is the confluent form of the Layman-Lotockiĭ series (1), (2).

The following more novel series, which is a generalization of (9), is obtained from the divided difference expansion of the operator e^{Dz} for $D = t_0, t_1, \ldots$:

$$f(z) = e^{Dz}f(0) = \left\{e^{t_0 z} + [(e^{t_1 z} - e^{t_0 z})/(t_1 - t_0)](D - t_0) + \cdots \right. \\ \left. + \left[\sum_{j=0}^{i} e^{t_j z}/\Pi^i_{k=0,k\neq j}(t_j - t_k)\right](D-t_0)\ldots(D-t_{i-1}) + \cdots\right\} f(0), \qquad (10)$$

converting a power series into an exponential series, its coefficients being linear combinations of "general factorial derivatives" $(D-t_0)\ldots(D-t_{i-1})f(0)$. For $t_j = \log r_j$, (10) is a series in r_j^z, of the same form as (3), (4), in which D and $\log r_j$ replace the Δ and $s_j = r_j - 1$.

That the first $n+1$ terms of (10) (this includes (9) as a special case), with their ith derivatives, at $z = 0$, is equal to $f^{(i)}(0)$, $i = 0(1)n$, follows from the quantities in brackets [. . .] becoming the successive divided differences of D^i considered as a quantity, for $D = t_j$, $j = 0(1)n$ (the $(i+1)$th bracket, and those after, vanishing), from which the expression within the braces $\{\cdots\}$, becoming the $(i+1)$-point Newton divided difference identity for the polynomial D^i, must reproduce D^i considered as an operator. q.e.d.

The uniqueness of the $(n+1)$-point completely osculatory exponential sum $\Sigma^n_{j=0} a_j e^{t_j z}$ obtained from (10) (and (9) too), is apparent from the determinant of $[\Sigma^n_{j=0} a_j e^{t_j z}]^{(i)}_{z=0}$, $i = 0(1)n$, which is the transposed Vandermondian $\|t_j^i\|$, $i, j = 0(1)n$.

CONCLUSION

In seeking new interpolation series, we have noted the heuristic value of employing different expansions of the same operational function (here it was $(1+\Delta)^z$ or e^{Dz}), even though at each step (here for $z = i$ or for the ith derivative) those two different expansions of that same operational function become identical (here the same polynomials in Δ or D), which preserves the interpolation property. This suggests further applications of this principle to other expansions of $E^z = (1+\Delta)^z = e^{Dz}$, in terms of different functions of the operators E, Δ or D, those in terms of polynomials (e.g., Legendre, Chebyshev, etc.) producing series that give either $f(i)$ for $z = i$, or $f^{(i)}(0)$ when differentiated i times and z set equal to zero, and those in terms of transcendental functions presenting interesting problems regarding their interpretation, convergence and utility, even when the series produced is not interpolatory.

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